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A Term Model for Synchronous Processes

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0. INTRODUCTION

A calculus for synchronous communicating systems has recently been formulated by Milner (1980b). The approach is similar to that in CCS (Milner, 1980a), where agents are taken to be words over an alphabet of operators. In the latter case the operators and their properties are chosen so as to reflect intuitions about how asynchronous processes interact. The more recent calculus is an attempt to formalise processes which interact in a time dependent or synchronous manner. The fundamental operator is *synchronous product*, $\times$, with which the new process $p \times q$ can be formed from processes p, q. The only actions which $p \times q$ can perform are of the form $p \times q \xrightarrow{a \cdot b} p' \times q'$, where p, q can perform the actions $p \xrightarrow{a} p'$, $q \xrightarrow{b} q'$, respectively. Thus $p \times q$ can proceed only by p and q performing actions hand in hand. By suitable interpretations of the operator $\cdot$ over actions $a \cdot b$ can be considered to be an actual communication or synchronisation between p and q or as the simultaneous occurrence of two distinct actions a and b.

Two delay operators δ, Δ are also used. For example, δp is process which performs exactly as p except that it may initially delay. This is formalised by allowing δp to have the action $\delta p \xrightarrow{1} \delta p$, where 1 is a distinguished action. The remaining operators, restriction and renaming are similar to those used in CCS.

So a *synchronous agent* can be taken to be a finite or infinite word over an alphabet of operators Σ, i.e., an element of CT_Σ. Any Σ-continuous partial order (for definition see Goguen *et al.* (1980)) determines a model by associating $\mathscr{I}_A(p)$ with the agent p, where $\mathscr{I}_A$ is the unique strict Σ-homomorphism from CT_Σ to A. In this paper we restrict our attention to a subclass of agents, namely, those which can be defined from the operators by using recursive definitions. We show that there exists a model I which is fully abstract with respect to a particular operational preorder $\precsim$, i.e., for recursively defined p, q, $p \precsim q$ if and only if $\mathscr{I}_I(p) < \mathscr{I}_I(q)$.

In Milner (1980) observational equivalence, $\sim$, between synchronous agents is defined and it is shown that asynchronous agents are precisely

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those which satisfy $p \sim \Delta p$. We modify this relation $\sim$ by taking into account the ability to diverge. In this way we obtain the operational preorder $\underset{\sim}{\sqsubset}$. This is similar to the approach used by Hennessy and Plotkin (1980), where an operational preorder was also obtained from the observational equivalence in Hennessy and Milner (1980) and Milner (1980).

The model I is of interest in that it is the initial Σ-cpo which satisfies a set of equations. These equations, together with Scott Induction would form a powerful proof system for synchronous systems.

The reader is assumed to be familiar with Milner (1980b) where the synchronous calculus and its possible usefulness is discussed at length. Nevertheless for completeness sake we define it and give its semantics in Section 1 and we define the operational preorder. The denotational semantics is given in Section 2 and for the sake of brevity we assume the reader is familiar with various notions used in algebraic semantics. This section also contains the statements of the main results. The following two sections are devoted to proofs. The final section, 5, contains a brief discussion of the results.

1. Calculus

The syntax is parameterised on the following sets:

(1) MV—a given countably infinite set of machine variables, ranged over by m.

(2) A—a given set of atomic actions. We assume that there is a binary operation, $\cdot$, over A and a distinguished element, 1, of A such that $\langle A, \cdot, 1\rangle$ is an abelian monoid.

Given these two sets we define the syntactic classes as follows:

(3) Res—the set of *restrictions*, ranged over by B, is the set of subsets of A which contain 1.

(4) Ren—the set of *renamings*, ranged over by Φ, is the set of endomorphisms of A.

(5) Term—the set of *terms*, ranged over by t, u, v, is given by $t ::= \phi \mid t+u \mid t \times u \mid a\colon t \mid \delta p \mid \Delta p \mid p \upharpoonright B \mid p[\Phi] \mid m \mid \mathrm{rec}\, m.t$.

Free and bound (machine) variables are defined in the usual way, rec m.—being the binding operator. A term is *closed* if it has no free variables and is open otherwise. Substitution of terms for free variables in a term $t[u_1 m_1,\ldots, u_k/m_k]$ is defined in the usual way. A term is *syntactically finite* if it contains no occurrence of a subterm of the form rec $m.t'$. An *instantiation* is a mapping from MV to the set of closed terms. We let γ range over instantiations and $t\gamma$ denote the result of substituting $\gamma(m)$ for each free occurrence of m in t. Note that $t\gamma$ is always closed. We let p, q, range over closed terms.

As in Hennessy and Milner (1980) and Milner (1980a, b), the operational semantics is specified by defining a relation $\xrightarrow{a}$, for each $a \in A$, over closed terms. Let $\xrightarrow{a}$ be the least relation over closed terms which satisfy

(1) $a{:}\,p \xrightarrow{a} p$.

(2) $p \xrightarrow{a} p'$, $q \xrightarrow{b} q'$ implies $p \times q \xrightarrow{a \cdot b} p' \times q'$.

(3) $p \xrightarrow{a} p'$ implies $p + q \xrightarrow{a} p'$, $q + p \xrightarrow{a} p'$.

(4) $\delta p \xrightarrow{1} \delta p$
$p \xrightarrow{a} p'$ implies $\delta p \xrightarrow{a} p'$.

(5) $p \xrightarrow{a} p'$ implies $\Delta p \xrightarrow{a} \delta\, \Delta p'$.

(6) $p \xrightarrow{a} p'$ implies $p \upharpoonright B \xrightarrow{a} p' \upharpoonright B$ if $a \in B$.

(7) $p \xrightarrow{a} p'$ implies $p[\Phi] \xrightarrow{\Phi(a)} p'[\Phi]$.

(8) $t[\mathrm{rec}\ m.t/m] \xrightarrow{a} p$ implies $\mathrm{rec}\ m.t \xrightarrow{a} p$.

We may now proceed as in Milner (1980a) to define the sequence of equivalence relations ~ 0, $\sim 1, \ldots$ over terms by examining their possible actions and finally obtain the equivalence $\sim$. Using the definitions of Milner (1980a) we obtain

$$a{:}\, \phi \sim \mathrm{rec}\ m.(a{:}\, \phi + m).$$

In any reasonable model based on Scott theory (Scott and Strachey, 1971), where recursively defined terms are interpreted as least solutions to fixpoint equations, these terms will not be identified. These models are based on partial orders, where intuitively p is considered less than q if it contains "less information" than q. So if we wish to relate the denotational models to operational models it is necessary to introduce an operational preorder. This is easily done by modifying the definition of $\sim$ so as to take into consideration divergence. We do this by defining a convergence predicate.

Let $\downarrow$ be the least relation over closed terms such that

(i) $a{:}\,p\downarrow$

(ii) $p\downarrow$ implies $\delta p\downarrow$
$\Delta p\downarrow$
$p[\Phi]\downarrow$
$(p \upharpoonright B)\downarrow$

(iii) $p\downarrow, q\downarrow$ implies $(p+q)\downarrow$, $(q+p)\downarrow$
$(p \times q)\downarrow$, $(q \times p)\downarrow$

(iv) $\phi\downarrow$

(v) $t[\mathrm{rec}\ m.t/m]\downarrow$ implies $\mathrm{rec}\ m.t\downarrow$.

Thus, if $p\downarrow$, then we can expand the recursive definitions on p a finite number of times to obtain at the top level all possible next moves of p. Let $\uparrow$ denote the complement of $\downarrow$, i.e., $p\uparrow$ if not $p\downarrow$.

EXAMPLE.

$$\text{rec } m.(a\colon \phi + b\colon m)\downarrow$$

$$\delta \text{ rec } m.(a\colon \phi + \delta\colon m)\uparrow$$

$$\delta \text{ rec } m.(a\colon \phi + 1\colon m)\downarrow$$

It is easy to prove that $p\uparrow$ if and only if there exists an infinite reduction sequence $p = p_0 \xrightarrow{e} p_1 \xrightarrow{e} \cdots$ where $p_n \xrightarrow{e} p_{n+1}$ if p_{n+1} can be obtained from p_n by expanding some recursive definition in p_n which is not contained in a subterm of the form $a\colon p'_n$.

We now define the operational preorder.

DEFINITION. For closed terms p, q

(i) $p \sqsubseteq_0 q$ for all p, q,

(ii) $p \sqsubseteq_{k+1} q$ if for every $a \in A$

(a) $p \xrightarrow{a} p'$ implies $\exists q' \cdot q \xrightarrow{a} q'$ and $p' \sqsubseteq_k q'$,

(b) $p\downarrow$ implies

(i) $q\downarrow$

(ii) $q \xrightarrow{a} q'$ implies $\exists p' \cdot p \xrightarrow{a} p'$ and $p' \sqsubseteq_k q'$.

(iii) $p \sqsubseteq q$ if for every $k \geqslant 0$ $p \sqsubseteq_k q$.

Note that $\sqsubseteq$ is reflexive and transitive but not necessarily antisymmetric. For example if p denotes $(\text{rec } m.m) + a\colon a\colon \text{rec } m.m$ and q denotes $(\text{rec } m.m) + (a\colon \text{rec } m.m) + a\colon a\colon \text{rec } m.m$, then $p \sqsubseteq q \sqsubseteq p$ but p and q are different.

PROPOSITION 1.1. *$p \sqsubseteq q$ if and only if for every context $C[\ ]$, $C[p] \sqsubseteq C[q]$.*

Proof. See Proposition 3 of Milner (1980b).

This relation can be extended to all terms by letting $t \sqsubseteq u$ if for every instantiation γ, $t\gamma \sqsubseteq u\gamma$. We also write $t \simeq u$ if $t \sqsubseteq u$ and $u \sqsubseteq t$.

2. DENOTATIONAL SEMANTICS

Let Σ be the alphabet of operators ϕ, $+$, $\times$, δ, Δ, $a\colon$ for each $a \in A$, $[\Phi]$ for each $\Phi \in \text{Ren}$, $\lceil B$ for each B in Res and the o-ary symbol Ω. For convenience we also extend the definition of term by allowing Ω to be a term. The operational semantics remain the same. This implies that $\Omega\uparrow$ and $\Omega \xrightarrow{a} p$ for no p, a. We assume that the reader is familiar with the notions of Σ-po, Σ-cpo, Σ-homomorphism, algebraicness, ideal completion, Σ-precongruence (called an F-magma preorder in Courcelle and Nivat (1976), etc. Details may be found in Courcelle and Nivat (1976) and Goguen *et al.* Let A be any Σ-cpo. A denotational semantics for the language is given by a mapping

$$\mathscr{V}_A\colon \mathrm{Term} \to [E_A \to A]$$

where E_A is the set of A-environments, i.e., $E_A = (\mathrm{MV} \to A)$. We let ρ range over E_A. As is usual $\rho[x/m]$ denotes the environment which is the same as ρ except at m, where it is defined to be x. For completeness sake we now define $\mathscr{V}_A$, by structural induction on terms:

(i) $\mathscr{V}_A[\![m]\!]\rho = \rho(m)$,

(ii) for any $\mathrm{op} \in \Sigma$ of arity $k \geqslant 0$

$$\mathscr{V}_A[\![\mathrm{op}(t_1,\dots, t_k)]\!]\rho = \mathrm{op}_A(\mathscr{V}_A[\![t_1]\!]\rho,\dots, \mathscr{V}_A[\![t_k]\!]\rho),$$

(iii) $\mathscr{V}_A[\![\mathrm{rec}\ m.t]\!]\rho = Y\lambda m' \cdot \mathscr{V}_A[\![t]\!]\,\rho[m'/m]v$, where Y denotes the least fixpoint operator.

Note that if t is closed, $\mathscr{V}_A[\![t]\!]\rho$ does not depend on ρ. For convenience we write $t <_A u$ if $\mathscr{V}_A[\![t]\!] < \mathscr{V}_A[\![u]\!]$ and $t =_A u$ if $t <_A u$ and $u <_A t$. We are looking for a particular A such that $t <_A u$ if and only if $t \sqsubseteq u$. Such an A is called *fully-abstract* w.r.t. the operational preorder $\sqsubseteq$. A natural choice of A would be the initial Σ-cpo, I_E, in the class of Σ-cpos which satisfy some set of equations E defined over the alphabet Σ. For example E might be a subset of the equations given in Milner (1980b). However no such E can exist.

EXAMPLE. $\delta a\colon \phi \sqsubseteq \mathrm{rec}\ m.(1\colon m + a\colon \phi)$. For any set of equations E, I_E is algebraic. Therefore if $\delta a\colon \phi <_{I_E} \mathrm{rec}\ m.(1\colon m + a\colon \phi)$ then there exists some finite term $d = 1\colon (1\colon (1\colon (\cdots(\Omega + a\colon \phi) + a\colon \phi) + \cdots + a\colon \phi)$ such that $\delta a\colon \phi <_{I_E} d$. Therefore $\delta a\colon \phi <_E d$ where $<_E$ is the least Σ-precongruence satisfying the axioms E. However if E is consistent then this is not possible since $\delta a\colon \phi \not\sqsubseteq d$.

We now state the main results of the paper. Let $\mathscr{C}$ denote the category of Σ-cpo's which satisfy the equations of Table 1 together with the law

$$\delta x = \mathrm{rec}\ m.(1\colon m + x) \qquad (*)$$

We take as morphisms strict continuous Σ-homomorphisms.

PROPOSITION 2.1. *$\mathscr{C}$ has an initial object I.*

TABLE 1

$(x+y)+z=x+(y+z)$ $x+y=y+x$ $x+x=x$ $x+\phi=x$		A
$x \times y = y \times x$ $(x_1+x_2)\times y = x_1 \times y + x_2 \times y$ $x \times \phi = \phi$ $a:x \times b:y = a \cdot b:(x \times y)$		S
$(a:x)\upharpoonright B = a:(x \upharpoonright B)$ $=\phi$ $(x+y)\upharpoonright B = x \upharpoonright B + y \upharpoonright B$ $a:x[\Phi] = \Phi(a):x[\Phi]$ $(x+y)[\Phi] = x[\Phi] + y[\Phi]$ $\phi[\Phi] = \phi \upharpoonright B = \phi$	if $a \in B$ otherwise	R
$\Delta(x+y) = \Delta x + \Delta y$ $\Delta a:x = a:\delta\Delta x$ $\Delta\phi = \phi$		A
$\Omega \times x = \Omega$ $\Omega \upharpoonright B = \Omega$ $\Omega[\Phi] = \Omega$ $\Delta\Omega = \Omega$		Ω

PROPOSITION 2.2. *I is fully-abstract* w.r.t. $\sqsubseteq$, *i.e.*, $t \sqsubseteq u$ *if and only if* $t <_I u$.

I is in fact an algebraic Σ-cpo and therefore it is the unique (up to isomorphism) algebraic fully-abstract model whose finite elements are definable by terms. Because of the law $(*)$ there does not appear to be any general theorem in the literature which can be invoked to obtain Proposition 2.1 as a corollary. However in the next section we show that I can easily be constructed.

There are many syntactically finite terms which are operationally equivalent but which cannot be proven so using the equations of Table 1. For example $\delta\delta a: \phi \eqsim \delta a: \phi$ but we cannot derive $\delta\delta a: \theta = \delta a: \phi$. This is because syntactically finite terms may be semantically infinite. Let $<_E$ denote the least Σ-precongruence generated by the equations of Table 1 and the equations

$$\Omega < x,$$

$$\delta x = 1: \delta x + x,$$

$$\delta\delta x = \delta x,$$

$$\begin{aligned}
\delta x \times \delta y &= \delta(x \times \delta y + \delta x \times y),\\
\delta x \times a\colon y &= x \times a\colon y + a\colon (\delta x \times y),\\
\Delta\, \delta x &= \delta\, \Delta x,\\
(\delta x)[\Phi] &= \delta(x[\Phi]),\\
(\delta x) \restriction B &= \delta(x \restriction B),\\
\delta\Omega &= \Omega
\end{aligned}$$

PROPOSITION 2.3. *For syntactically finite closed terms p, q $p \underset{\sim}{\sqsubset} q$ if and only if $p <_E q$.*

It follows that these equations and some form of Scott Induction would form the basis of a proof system for synchronous communicating systems.

The remaining two sections are devoted to the proofs of these three propositions.

3. CONSTRUCTION OF THE INITIAL MODEL

We first recall some relevant facts from Courcelle and Nivat (1976). Let A, B be Σ-algebras and $<$ any Σ-precongruence over A. A Σ-homomorphism $h\colon A \to B$ preserves $<$ if $a_1 < a_2$ implies $h(a_1) <_B h(a_2)$. Then given any Σ-algebra A and Σ-precongruence over A, there exists a unique Σ-partial order $\langle (A/<), < \rangle$ such that

(i) there is a Σ-homomorphism

$$i\colon A \to (A/<)$$

(ii) given any Σ-homomorphism $h\colon A \to B$, where B is a Σ-po, which preserves $<$, there exists a unique extension $(h/<)\colon (A/<) \to B$ which is a monotonic Σ-homomorphism, and such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & B \\
(A/<) \arrow[ur, "(h/<)"'] &
\end{tikzcd}$$

The partial order $\langle (A/<), > \rangle$ is just the set of equivalence classes generated by the Σ-precongruence $<$ and ordered by the relation $[a] < [a']$ if $a < a'$.

Let $\langle A, <, \perp \rangle$ be a Σ-partial order $\langle A, < \rangle$ together with a least element $\perp \in A$, w.r.t. $<$. Then there exists a unique Σ-cpo$\langle A^{\infty}, <^{\infty}, \perp^{\infty} \rangle$ such that

(i) there exists a strict monotonic Σ-homomorphism

$$i: A \to A^{\infty}$$

(ii) given any monotonic Σ-homomorphism $h: A \to B$ where B is a Σ-cpo, there exists a unique extension $h^{\infty}: A^{\infty} \to B^{\infty}$ which is a continuous Σ-homomorphism such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & B \\ A^{\infty} \arrow[ur, "h^{\infty}"'] & \end{tikzcd}$$

The Σ-cpo A^{∞} is called the *ideal completion* of A. It is obtained by ordering the set of ideals of A by set inclusion. Details may be found in Courcelle and Nivat (1976).

Let Σ^0 be the alphabet $\Sigma \backslash \{\Delta, \delta\}$. Let I^0 be the initial object in the category of Σ^0-cpo's which satisfy the equations (A, S, R, Ω) of Table 1. I^0 can be described algebraically as $(W_{\Sigma^0}/<_1)^{\infty}$ where W_{Σ^0} is the word algebra over the alphabet Σ^0 and $<_1$ is the least Σ^0-precongruence over W_{Σ^0} which satisfies the equations (A, S, R, Ω) and the equation $\Omega < x$. I^0 can be given a relatively concrete representation in terms of trees. Let T be the set of finite and infinite, rooted, unordered finitely branching trees, each of whose arcs is labelled by a member of A. The operator a_T takes a tree t and transforms it into the tree whose root has a unique arc coming from it which is labelled by a and whose descendant is t. The operator $+_T$ simply joins two trees at their roots. The remaining operators are defined recursively so as to ensure that the equations (S, R, Ω) are satisfied. The resulting Σ^0-cpo called T, is not exactly I^0 since it does not satisfy the equation $x + x = x$. But I^0 can be obtained from T by factoring out with respect to this equation. This rather imprecise description may help the reader to visualise I^0 but we will use the algebraic characterisation $(W_{\Sigma^0}/<_1)^{\infty}$.

We now extend I^0 to a Σ-cpo I. The carrier of I is exactly that of I^0 so it is sufficient to define two functions over it, δ_I, Δ_I.

(i) Definition of δ_I:

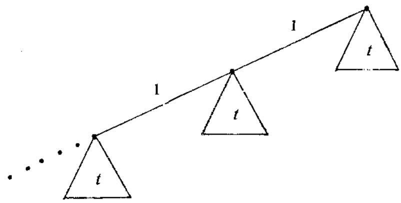
Define $\delta_S: W_{\Sigma^0_\Omega} \to I^0$ by $\delta_S(p) = \mathscr{V}_{I^0}[\![\text{rec } m.(1: m + p)]\!]$.

LEMMA 3.1. *δ_S preserves $<_1$.*

Proof. It follows immediately from the fact that I^0 satisfies the axioms (A, S, R, Ω).

Let δ_I be $(\delta_S/<_1)^{\infty}$.

Using the informal description of I^0 presented above δ_I maps the tree t to the tree:



(ii) Definition of Δ_I:

LEMMA 3.2. *For any $p \in W_{\Sigma^0}$ there exists a normal form $nf(p) \in W_{\Sigma^1}$, $\Sigma^1 = \Sigma^0 \backslash \{\times, [\Phi], \upharpoonright B\}$ such that $p =_1 nf(p)$.*

Proof. By structural induction on p. It is sufficient to eliminate the operators $\times$, $[\Phi]$, $\upharpoonright B$, using the axioms (A, S, R, Ω).

Thus every finite element in the model is equivalent to a term over the alphabet $\Sigma^1 = \{\Omega, a\!:, +\}$. This fact will be used frequently in the sequel. So we let d range over elements of W_{Σ^1}.

Define $\Delta_S: W_{\Sigma^1} \to I^0$ by structural induction on normal forms:

(i) $\Delta_S(\Omega) = \Omega$,

(ii) $\Delta_S(\phi) = \phi$,

(iii) $\Delta_S(a\!:p) = a_I(\delta_I(\Delta_S\, p)))$,

(iv) $\Delta_S(p_1 + p_2) = \Delta_S(p_1) +_I \Delta_S(p_2)$.

This automatically defines Δ_S on all finite terms by letting $\Delta_S(p)$ be $\Delta_S(nf(p))$. It is easy to show that Δ_S preserves $<_1$. Therefore we can define Δ_I to be $(\Delta_S/<_1)^\infty$.

Let I be the Σ-cpo just described.

PROPOSITION 3.3 (Proposition 2.1). *I is initial in the category of Σ-cpo's satisfying the equations of Table 1 and the law $(*)$.*

Proof. By construction I satisfies all the equations of Table 1 and the law $(*)$. Also any strict continuous Σ^0-homomorphism from I^0 to D is also a strict continuous Σ-homomorphism. If $D \in \mathscr{C}$, D is also in the category of Σ^0-cpo's which satisfy the laws (A, S, R, Ω). Let h_D be the unique strict continuous Σ^0-homomorphism from I^0 to D. Then h_D is also a Σ-

homorphism. It is unique in the category since all Σ-homomorphisms are determined by Σ^0-homomorphisms.

In the remainder of this section we examine the relation $<_I$ in more detail.

Let $\sim$ be the least Σ-congruence over closed terms which satisfies $\operatorname{rec} m.t = t[\operatorname{rec} m.t/m]$ for every term t. Let p_Ω be the finite term obtained from p by substitting Ω for every occurrence of $\operatorname{rec} m.t$. Then the *syntactic approximants* to p is defined to be $SA(p) = [q_\Omega \mid q \sim p\}$.

LEMMA 3.4. $\mathscr{V}_A[\![p]\!] = \bigvee_A SA(p)$.

Proof. A standard result of algebraic semantics. For details see Goguen *et al.* and Nivat (1975).

These syntactic approximants may include terms which are semantically infinite. We now define the set of *semantic approximants* to terms. Because of Lemma 3.4 it is sufficient to consider syntactically finite terms. For every $p \in W_\Sigma$ and every n, we define $a_n(p) \in W_{\Sigma 1}$, the nth semantic approximant to p, by induction on p:

(i) $a_0(p) = \Omega$ for every p;

(ii) $a_{n+1}(\Omega) = \Omega$, $a_{n+1}(\phi) = \phi$,

$$a_{n+1}(a : p) = a : a_{n+1}(p),$$
$$a_{n+1}(p_1 + p_2) = a_{n+1}(p) + a_{n+1}(p),$$
$$a_{n+1}(\delta p) = 1 : a_n(\delta p) + a_n(p);$$

(iii) let $a_{n+1}(p)$ be $\sum_{i \in I} a_i : p_i$; then

$$a_{n+1}(p \upharpoonright B) = \sum_{i \in I_0} a_i : a_n(p_i \upharpoonright B) \qquad \text{where} \quad I_0 = \{i \mid a_i \in B\},$$
$$a_{n+1}(p[\Phi]) = \sum_{i \in I} \Phi(a_i) : a_n(p_i[\Phi]),$$
$$a_{n+1}(\Delta p) = \sum_{i \in I} a_i : a_n(\delta\, \Delta p_i);$$

(iv) If $a_{n+1}(p)$ is $\sum_{i \in I} a_i : p + \Omega$, then Δp, $p[\Phi]$, $p \upharpoonright B$ are defined in a similar fashion. We merely add the extra summand Ω.

In this definition we have used $\sum_{i \in I} p_i$ to denote the term $p_{i_1} + (p_{i_2} + \cdots + p_{i_n}) \cdots)$ where I is the finite set $\{i_1, \ldots, i_n\}$. If I is empty, it denotes ϕ.

LEMMA 3.5. *For finite p, $\mathscr{V}_I[\![p]\!] = \bigvee \{\mathscr{V}_I(a_n(p) \mid n \geqslant 0\}$.*

Proof. By structural induction on p. We examine two cases.

(i) p is δq. Let $F: I \to I$ denote the function $\lambda p.1_I(p) +_I p$. Then

$$\begin{aligned}
\mathscr{V}_I(\delta q) &= \bigvee \{F^n(q) \mid n \geqslant 0\} && \text{where } F^n \text{ is the } n\text{th iterate of } F \\
&= \bigvee \{F^n(a_n(q)) \mid n \geqslant 0\} && \text{by induction and continuity} \\
&= \bigvee \{1_I(a_n(q)) +_I a_n(q) \mid n \geqslant 0\} \\
&= \bigvee \{\mathscr{V}_I(a_{n+1}(\delta q)) \mid n \geqslant 0\}.
\end{aligned}$$

(ii) p is Δq. By construction $\mathscr{V}_I(a_{n+1}(\Delta q)) = \Delta_I \mathscr{V}_I(a_n(q))$. Therefore,

$$\begin{aligned}
\mathscr{V}_I(\Delta q) &= \Delta_I(\bigvee \{\mathscr{V}_I(a_n(q)) \mid n \geqslant 0\}) && \text{by induction} \\
&= \bigvee \{\Delta_I \mathscr{V}_I(a_n(q)) \mid n \geqslant 0\} \\
&= \bigvee \{\mathscr{V}_I(a_{n+1}(\Delta q) \mid n \geqslant 0\}.
\end{aligned}$$

Let $<_2$ be the least Σ-precongruence over closed terms which satisfy the equations of Table 1 and

$$\Omega < x \tag{E2.1}$$

$$\mu m.t = t[\mu m.t/m] \qquad \text{for all } t \tag{E2.2}$$

$$\delta x = 1\colon \delta x + x. \tag{E2.3}$$

PROPOSITION 3.6. *$d <_I p$ if and only if $d <_2 p$.*

Proof. Suppose $d <_I p$. From Lemma 3.4, $\mathscr{V}_I[\![p]\!] = \bigvee SA(p)$. Since d is a finite element in I, $\exists q \sim p$ such that $d <_I q_\Omega$. Now $q_\Omega <_2 p$, using equations (E2.1), (E2.2). Also applying Lemma 3.5 to q_Ω we get an n s.t. $d <_I a_n(q_\Omega)$. Since both are finite $d <_1 a_n(q_\Omega)$. It follows that $d <_2 p$ since $a_n(r) <_2 r$ for any r. This can be proven by induction on n and the structure of r. To show the converse it is sufficient to show that $<_I$ satisfies the equations on which $<_2$ is based. This is true by the construction of I.

4. ANALYSIS OF THE OPERATIONAL PRE-ORDER

For any relation R over closed terms define $R^{\mathscr{C}}$ by $\langle p, q\rangle \in R^{\mathscr{C}}$ if $(\forall d \in W_{\Sigma^1}.\langle d, p\rangle \in R$ implies $\langle d, q\rangle \in R)$. R is *bfa* (behaviourally finitely approximable) if $R = R^{\mathscr{C}}$.

EXAMPLE. The relation $<_I$ is bfa. This follows since I is an algebraic cpo with all the finite elements denotable by elements of W_{Σ^1}.

We now show that $\lesssim$ is also bfa. Obviously $\lesssim \subseteq \lesssim^{\mathscr{C}}$. To show the converse we introduce another set of approximants. Let $F \subseteq A$ be a finite set and T_F be the set of closed terms which contain no occurrences of any operator a: where $a \notin F$. Define $A^F(0) = \{\Omega\}$. Assume that there exists a finite set $A^F(n) \subseteq W_{\Sigma^1}$, such that for every $p \in T_F$ there exists some element $p^{(n)} \in A^F(n)$ such that $p \lesssim_n p^{(n)}$ and $p^{(n)} \lesssim_n p$. Let

$$A^F(n+1) = \left\{ \sum_{i \in I} p_i \mid I \text{ is finite, } p_i \text{ is } \Omega \text{ or } a{:}\, d, \text{ where } a \in F,\ d \in A^F(n), \right.$$
$$\left. i \neq j \text{ implies } p_i \neq p_j \right\}.$$

Note that $A_F(n+1) \subseteq W_{\Sigma^1}$ and is finite. For any p let $p^{(n+1)} = \sum \{a{:}\, p_1^{(n)} \mid p \xrightarrow{a} p'\} + \{\Omega \mid p\uparrow\}$. Note that $p^{(n+1)}$ is well-defined, even if it has an infinite number of derivatives, since $A^F(n)$ is finite. It is simple to show $p^{(n+1)} \in A^F(n+1)$ and $p^{(n+1)} \lesssim_{n+1} p$, $p \lesssim_{n+1} p^{(n+1)}$.

LEMMA 4.1. $\lesssim$ is bfa.

Proof. Supose $p \lesssim^{\mathscr{C}} q$. Then for every $n \geqslant 0$, $p^{(n)} \lesssim q$. It follows that $p \lesssim_n p^{(n)} \lesssim q$, i.e., $p \lesssim_n q$ for every $n \geqslant 0$, and therefore $p \lesssim q$.

Let D be the defining functional of $\lesssim$, i.e., $\langle p, q\rangle \in D(R)$ if for every $a \in A$

(i) $p \xrightarrow{a} p'$ implies $\exists q' \cdot q \xrightarrow{a} q'$ and $\langle p', q'\rangle \in R$;

(ii) $p\downarrow$ implies

  (a) $q\downarrow$

  (b) $q \xrightarrow{a} q'$ implies $\exists p' \cdot p \xrightarrow{a} p'$ and $\langle p', q' \in R$.

Let $\lesssim^m$ be the maximal fixpoint of the equation $R = D(R)$. Now $\lesssim^m \subseteq \lesssim$ but the converse is not necessarily true. We do however have a partial converse.

LEMMA 4.2. *$d \lesssim p$ implies $d \lesssim^m p$.*

Proof. Define the depth of d, $dt(d)$ as follows:

(i) if $\nexists a\ \nexists d'.d \xrightarrow{a} d'$, then $dt(d) = 1$;

(ii) otherwise $dt(d) = 1 + \max\{dt(d') \mid \exists a \cdot d \xrightarrow{a} d'\}$.

Note that $dt(d)$ is well-defined since d is always in W_{Σ^1}. Then it is a simple matter to show, by induction on $dt(d)$, that $d \lesssim_{dt(d)} p$ implies $d \lesssim^m p$.

For the remainder of this section we will identify $\lesssim$ and $\lesssim^m$ when applied to $\langle d, p\rangle$. Let $<_3$ be the least Σ-precongruence which contains $<_2$ and which satisfies the equation

$$a{:}\, x \times (y + b{:}\, z) = a{:}\, x \times (y + b{:}\, z) + a \cdot b(x \times z)$$

Let $p \sim_3 q$ if $p <_3 q$ and $q <_3 p$.

LEMMA 4.2. *$p \xrightarrow{a} p'$ implies $p + a : p' \sim_3 p$.*

Proof. By induction on why $p \xrightarrow{a} p'$. We examine three cases. The remainder are similar.

(i) p is δq and $q \xrightarrow{a} p'$. By induction $q + a : p' \sim_3 q$. Therefore

$$\begin{aligned} p + a : p' &\sim_3 1 : p + q + a : p' \\ &\sim_3 1 : p + q \\ &\sim_3 p. \end{aligned}$$

(ii) p is $\mathrm{rec}\, m.t$ and $t[\mathrm{rec}\, m.t/m] \xrightarrow{a} p'$. By induction $t[\mathrm{rec}\, m.t/m] + a : p' \sim_3 t[\mathrm{rec}\, m.t/m]$. Therefore

$$\begin{aligned} p + a : p' \sim_3 t[\mathrm{rec}\, m.t/m] + a : p &\sim_3 t[\mathrm{rec}\, m.t/m] \\ &\sim_3 p. \end{aligned}$$

(iii) p is $p_1 \times p_2$, $p_1 \xrightarrow{a_1} p_1'$, $p_2 \xrightarrow{a_2} p_2'$, p' is $p_1' \times p_2'$ and a is $a_1 \cdot a_2$. By induction $p_i + a_i : p_i' \sim_3 p_i$, $i = 1, 2$. Therefore

$$\begin{aligned} p &\sim_3 (p_1 + a_1 : p_1') \times p_2 \\ &\sim_3 p_1 \times p_2 + a_1 : p_1' \times (p_2 + a_2 : p_2') \\ &\sim_3 p_1 \times p_2 + a_1 : p_1' \times (p_2 + a_2 : p_2') + a_1 \cdot a_2 : p_1' \times p_2') \\ &\sim_3 p + a : p'. \end{aligned}$$

For syntactically finite terms we already have the notion of a normal form. This is now generalised to infinite terms by defining head normal forms. A closed term is in *head normal form*, hnf, if it has the form $\sum_{i \in I} a_i : p_i$.

LEMMA 4.4. *$p\downarrow$ implies $\exists$ a hnf, $\mathrm{hnf}(p)$ such that $p \sim_3 \mathrm{hnf}(p)$.*

Proof. By induction on why $p\downarrow$. For example, if p is δq, then $q\downarrow$. By induction $\mathrm{hnf}(q)$ exists. Let $\mathrm{hnf}(\delta q)$ be $1 : \delta q + \mathrm{hnf}(q)$. The remaining cases are similar.

PROPOSITION 4.5. *$d \sqsubseteq p$ if and only if $d <_3 p$.*

Proof. Suppose $d <_3 p$. Since $\sqsubseteq$ is a Σ-congruence to show $d \sqsubseteq p$ it is sufficient to prove $\sqsubseteq$ satisfies all the equations on which $<_3$ is based. We leave this to the reader.

Conversely suppose $d \sqsubseteq p$. We prove by induction on d that $d <_3 p$. The proof is divided into two parts:

(a) $d <_3 d + p$. If $d\uparrow$ then $d \sim_3 d + \Omega$, and therefore $d \sim_3 d +$

$\Omega <_3 d+p$. If $d\downarrow$, then $p\downarrow$. Let hnf(p) be $\Sigma a_i : p_i$. Then $p \xrightarrow{a_i} p_i$. So $\exists d'$ such that $d \xrightarrow{a} d'$ and $d' \sqsubseteq p_i$. By induction $d' <_3 p_i$. Therefore, $d \sim_3 d + a_i : d' <_3 d + a_i : p_i$. Since this is true for each i it follows that $d <_3 d + p$.

(b) $d + p <_3 p$. Let d be $\Sigma a_i d_i$. As in part (i) for every i there exists a p_i such that $p \xrightarrow{a_i} p_i$ and $d_i \sqsubseteq p_i$. By induction $d_i <_3 p_i$. Therefore $d + p <_3 \Sigma a_i : p_i + p \sim_3 p$.

COROLLARY 4.6 (Proposition 2.2). $t \sqsubseteq u$ if and only if $t <_I u$.

Proof. The model I satisfies all of the equations on which $<_3$ is based. It follows that $p <_3 q$ implies $p <_I q$. From Proposition 3.6, $d <_3 q$ if and only if $d <_2 q$. Applying Propositions 3.6 and 4.5 we get $d \sqsubseteq q$ if and only if $d <_I q$. Since both relations are bfa, it follows that $p \sqsubseteq q$ if and only if $p <_I q$.

It remains to show the result for open terms. For any instantiation γ let ρ_γ be the I-environment defined by $\rho_\gamma(m) = \mathscr{T}_I[\![\gamma(m)]\!]\,\rho'$. Then we can show $\mathscr{T}_I[\![t\gamma]\!]\,\rho' = \mathscr{T}_I[\![t]\!]\,\rho_\gamma$, for any ρ'. Now suppose $t <_I u$. Then for any instantiation γ, $\mathscr{T}_I[\![t]\!]\,\rho_\gamma = \mathscr{T}_I[\![t\gamma]\!]\rho' < \mathscr{T}_I[\![u\gamma]\!]\,\rho'$. Since both $t\gamma$ and $u\gamma$ are closed it follows that $t\gamma \sqsubseteq u\gamma$.

Conversely suppose $t \sqsubseteq u$. An I environment ρ is *finite* if every $\rho(m)$ is a finite element in I. Since I is algebraic to show $t <_I u$, it is sufficient to show that $\mathscr{T}_I[\![t]\!]\rho < \mathscr{T}_I[\![u]\!]\rho$ for finite ρ. However, if ρ is finite there exists an instantiation γ such that $\rho = \rho_\gamma$. Since $t\gamma \sqsubseteq u\gamma$ and both are closed $\mathscr{T}_I[\![t\gamma]\!]\,\rho' < \mathscr{T}_I[\![u\gamma]\!]\,\rho'$. As above it follows that $\mathscr{T}_I[\![t]\!]\rho < \mathscr{T}_I[\![u]\!]\rho$.

We now concentrate on Proposition 2.3. Let $\Sigma^2 = \Sigma \backslash \{[\Phi], \restriction B, \times, \Delta\}$.

LEMMA 4.7. *For every p in W_Σ there exists a term $\delta nf(p)$ in W_{Σ^2} such that $p \sim_E \delta nf(p)$.*

Proof. The operators $[\Phi]$, $\restriction B$, $\times$ and Δ can be eliminated by applying the equations from left to right. The details are left to the reader. As usual $\sim_E$ denotes $<_E \cap_E >$.

LEMMA 4.8. *If $p \in W_{\Sigma^2}$ and $p\uparrow$, then $p \sim_E p + \Omega$.*

Proof. By structural induction on p. We examine two cases.

(i) p is δq. Then $q\uparrow$. By induction $q + \Omega \sim_E q$. Therefore

$$\begin{aligned} p &\sim_E 1 : \delta q + q \\ &\sim_E 1 : \delta q + q + \Omega \\ &\sim_E p + \Omega. \end{aligned}$$

(ii) p is $p_1 \times p_2$. Without loss of generality suppose $p_1 \uparrow$. By induction $p_1 + \Omega \sim_E p_1$. Therefore

$$\begin{aligned} p &\sim_E (p_1 + \Omega) \times p_2 \\ &\sim_E p_1 \times p_2 + \Omega \times p_2 \\ &\sim_E p_1 \times p_2 + \Omega. \end{aligned}$$

LEMMA 4.9. *If $p \in W_{\Sigma^2}$ and $p \xrightarrow{a} p'$, then $p \sim_E p + a:p$.*

Proof. Similar to lemma 4.3. We examine one case only, when p has the form δq.

(a) p' is δq and a is 1. Then $p \sim_E 1:\delta q + q \sim_E 1:\delta q + q + 1:\delta q \sim_E p + 1:p'$

(b) $\xrightarrow{a} p'$. By induction $q \sim_E q + a:p'$. Therefore

$$\begin{aligned} p &\sim_E 1:\delta q + q \\ &\sim_E 1:\delta q + q + a:p' \\ &\sim_E p + a:p'. \end{aligned}$$

Lemma 4.10. *For $p, q \in W_{\Sigma^2} p \sqsubseteq q$ if and only if $p \sqsubseteq^m q$.*

Proof. The relations $\xrightarrow{a}$ when restricted to W_{Σ^2} are image-finite, i.e., for every $p \in W_{\Sigma^2}$, $\{p' \mid p \xrightarrow{a} p'\}$ is finite. The result then follows from (Hennessy and Milner, 1980).

LEMMA 4.11. *$p\downarrow$, $\delta p \sqsubseteq \delta q$ implies $p \sqsubseteq \delta q$, $\delta p \sqsubseteq q$* or *$p \sqsubseteq q$.*

Proof. (a) Suppose that $p \xrightarrow{1} p'$ implies $p' \not\sqsubseteq \delta q$. There are two subcases:

(i) $q \xrightarrow{1} q'$ implies $\delta p \not\sqsubseteq q'$. It follows that $p \sqsubseteq q$.

(ii) $\exists q' \cdot q \xrightarrow{1} q'$ such that $\delta p \sqsubseteq q'$. In this case $\delta p \sqsubseteq q$.

(b) Suppose that $q \xrightarrow{1} q$ implies $\delta p \not\sqsubseteq q'$. The only possibility not considered is when there exists a p' such that $p \xrightarrow{1} p'$ and $p' \sqsubseteq \delta q$. In this case $p \sqsubseteq \delta q$.

PROPOSITION 4.12 (Proposition 2.3). *If p, q are syntactically finite and closed then $p \sqsubseteq q$ if and only if $p <_E q$.*

Proof. The relation $\sqsubseteq$ satisfies all the equations and since it is a Σ-precongruence, it follows that $p <_E q$ implies $p \sqsubseteq q$. To prove the converse we show

(a) $p \sqsubsetapprox p+q$ implies $p <_E p+q$,

(b) $p+q \sqsubsetapprox q$ implies $p+q <_E q$,

(c) $p \sqsubsetapprox q$ implies $p <_E q$.

We use induction on $\langle p, q \rangle$ and from Lemma 4.7 we can assume that both are in $W_{\Sigma 2}$.

(a) If $p\uparrow$, then $p \sim_E p + \Omega <_E p + q$. Otherwise we can assume $q\downarrow$. Let $q = \sum_{j \in J} q_j$. We show $p <_E p + q_j$ for each $j \in J$.

(i) q_j is of the form $a : r$. Then there exists a p' such that $p' \sqsubsetapprox r$ and $p \xrightarrow{a} p'$. Then

$$\begin{aligned} p &<_E p + a : p' \\ &<_E p + a : r \text{ by induction.} \end{aligned}$$

(ii) q_j is of the form δr. Now $p \sqsubsetapprox p + r$ so by induction $p <_E p + r$. Also since $q \xrightarrow{1} q_j$ there exists some p' such that $p \xrightarrow{1} p'$ and $p' \sqsubsetapprox q$. If either p' or q_j' are smaller than p, q, respectively, then we can apply induction to obtain

$$p \sim_E p + 1 : p' <_E p + r + 1 : q_j <_E p + q_j.$$

Otherwise p, q are of the form $\delta p''$, δr, respectively, and $\delta p'' \sqsubsetapprox \delta r$. We apply Lemma 4.11 and no matter what the conclusion is we can apply induction. For example if $p'' \sqsubsetapprox \delta r$ we get $p'' <_E \delta r$ and therefore $\delta p'' <_E \delta p'' + \delta p'' <_E \delta \delta r \sim_E \delta p'' + \delta r$.

(b) Let $p = \sum_{j \in J} p_j$. Each p_j is of the form Ω, $a : p'$ or $\delta p'$. Using the same techniques as in (a) we can show $p_j + q <_E q$ for each $j \in J$.

(c) follows immediately from (a), (b).

5. Discussion

The existence of a fully abstract term model for an operational semantics does not a priori lead to a better understanding of the objects being modelled. For example, the model of PCF, given in Milner (1977) is not very informative on the nature of sequential functions.

Fortunately our term model is rather simple and easy to understand. The objects are (approximately) finite and infinite trees, whose branches are labelled by actions. The operators a: and $+$ are defined in a very straightforward fashion on these trees. The remaining operators are not semantically significant since they can be defined in terms of a: and $+$. We have shown how to construct this model by essentially factoring the syntactic cpo of all finite and infinite terms by the axioms given in Table 1. This construction in itself has considerable significance since it gives us a complete axiomatic proof system for closed terms of the language. This proof system consists of:

—the axioms from Table 1
—the axiom $\Omega < X$
—rules which state $<$ is reflexive and transitive
—the rules

$$\frac{x < y, y < x}{x = y}, \qquad \frac{x = y}{x < y, y < x}$$

—the rules $x < y$ implies $C[x] < C[y]$ for every context $C[\ ]$
—the rule

$$\frac{\forall f \in SA(p), f < q}{p < q}.$$

This proof system is not recursively enumerable because of the last rule. However a complete recursively enumerable axiomatisation cannot exist, as was pointed out in Milner (1982), and so our system is as good as can be expected. To obtain r.e. proof systems the last rule may be replaced by Scott Induction or Fixpoint Induction, which it clearly implies. Although these systems may not be useful in practice they can form the basis of more realistic proof systems over derived languages.

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References

Courcelle, B., and Nivat, M. (1976), "Algebraic Families of Interpretations," IRIA Rapport de Recherche No. 189, 1976.

Goguen, J. A., Thatcher, J. W., Wagner, E. G., and Wright, J. B., Initial algebra semantics and continuous algebras, *J. Assoc. Comput. Mech.* **24,** No. 1, 68–95.

Hennessy, M., and Milner, R. (1980), "On Observing Nondeterminism and Concurrency, Lecture Notes in Computer Science No. 85, Springer-Verlag, Berlin/Heidelberg/New York.

Hennessy, M., and Plotkin, G. (1980), "A Term Model for CCS," Lecture Notes in Computer Science No. 88, Springer-Verlag, Berlin/Heidelberg/New York.

MILNER, R. (1977), Fully abstract models of typed λ-calculi, *Theoret. Comp. Sci.*, 4, No. 1, 1–23.

MILNER, R., (1980a), "A Calculus of Communicating Systems," Lecture Notes in Computer Science, No. 92, Springer-Verlag, Berlin/Heidelberg/New York.

MILNER, R. (1980b), "On Relating Synchrony and Asynchrony," CSR-75-80, University of Edinburgh.

MILNER, R. (1982), "Calculi for Synchrony and Asynchrony," CSR-104-82, University of Edinburgh.

NIVAT, M. (1975), On the interpretation of polyadic recursive schemes, *in* "Symposia Mathematica," Vol. 15, Academic Press, New York.

SCOTT, D., AND STRACHEY, C. (1971), "Towards a Mathematical Semantics for Computer Languages, Proc. Symposium on Computers and Automata," Microwave Institute Symposia Series 21, Polytechnic Institute of Brooklyn, Brooklyn.